9.6: The Logic of Relations: Proofs

The inference rules for predicate logic do not change when we extend our language to include relations. Nonetheless, some of the restrictions on the rules become particularly pertinent in the new proof contexts that can arise when relations are allowed. Hence, a number of reminders are in order.

1. If you have a premise with more than one quantifer, apply UI or EI to remove the quantifiers one at a time, from left to right.

Example (done in class):

1. $(x)(y)(Hxy \rightarrow \sim Ia)$
2. $(\exists x)Hbx \bullet (x)(y)(z)Jxyz \quad \therefore \; \sim(Jabc \rightarrow Ia)$

2. Remember that the generalization and instantiation rules (EG, UG, EI, UI) are implicational rules, not equivalence rules.

In the case of UI and EI, be sure that the quantifiers you apply these rules to have scope over the entire statement in a line of a proof, not just part of it.

1.	$(\exists x)(y)Fxy \rightarrow Gb$		
2.	$(x)(y)Fyx$	∴ Gb	
3.	$(y)Fay \rightarrow Gb$	1, EI	(***Incorrect!***)
4.	$Fab \rightarrow Gb$	3, UI	(***Incorrect!***)
5.	$(y)Fyb$	2, UI	
6.	Fab	5, UI	
7.	Gb	4,6 MP	

The *argument* above is in fact invalid, it just hasn't been *proved* correctly. Here is a correct proof (done in class):

1.	$(\exists x)(y)Fxy \rightarrow Gb$	
2.	$(x)(y)Fyx$	∴ Gb

In the case of EG and UG, be sure be sure the rules are applied to the entire statement in a line of a proof.

1. $(x)Ax \rightarrow Bac$ $\therefore (x)Ax \rightarrow (\exists y)Byc$
2. $(x)Ax \rightarrow (\exists y)Byc$ 1, EG (***Incorrect!***)

Done correctly (in class):

1. $(x)Ax \rightarrow Bac$ $\therefore (x)Ax \rightarrow (\exists y)Byc$

Consider now UG:

1. $(x)[(y)Lxy \rightarrow Ma]$ $\therefore (x)(y)Lxy \rightarrow Ma$
2. $(y)Lby \rightarrow Ma$ 1, UI
3. $(x)(y)Lxy \rightarrow Ma$ 2, UG (***Incorrect!***)

Done correctly (in class):

1. $(x)[(y)Lxy \rightarrow Ma]$ $\therefore (x)(y)Lxy \rightarrow Ma$

3. When applying UI and EI, remember that constants must be substituted for variables UNIFORMLY.

1. (x)[Mx • (Lx ∨ (y)Kxy)]
2. Mb • (Lb ∨ (y)Kby) 1, UI (*Correct*)
3. Ma • (Lb ∨ (y)Kby) 1, UI (***Incorrect!***)

Similarly for EI:

1. (∃x)(y)(Pxy ↔∼Oxy)
2. (y)(Pay ↔ ∼Oay) 1, EI (*Correct*)
3. (y)(Pby ↔ ∼Ocy) 1, EI (***Incorrect!***)

4. Remember, when applying EI, to choose a constant that has not occurred previously in the proof.

1. (x)(∃y)Gyx
2. (∃y)Gya 1, UI
3. Gaa 2, EI (***Incorrect!***)
4. (∃x)Gxx 3, EG

To allow this argument form would be to sanction arguments like the following: "Everyone has a mother (i.e., for every x, there is a y such that y is the mother of x). Therefore, someone is his or her own mother." Or again: "For every number there is a larger number. Therefore, some number is larger than itself."

5. Do not overlook the special restrictions on UG!

UG allows us to infer a universal generalization $(x)\mathcal{P}$ from $\mathcal{P}_c$ so long as the constant c does not occur in:

a. $(x)\mathcal{P}$ itself, or
b. a premise of the argument, or
c. a line derived by an application of EI, or
d. an undischarged assumption.

Consider the following:

1.	(y)Eyy	
2.	Ebb	1, UI
3.	(x)Exb	2, UG (***Incorrect!*** *Violation of condition a*)
4.	(∃y)(x)Exy	3, EG

We can see intuitively that violation here leads to incorrect results, as it sanctions such arguments as:

Everyone is as tall as themselves. Therefore, someone is as tall as everyone.

Preventing such inferences is the purpose of restriction *a* above: When you generalize on a constant c in a formula $\mathcal{P}_c$, you must generalize upon *every* occurrence of c in $\mathcal{P}_c$

Thus, in particular, the only permissible generalization (involving the variable ‘x’) that you can derive from line 2 by UG are (x)Exx.

As a final example, recall a somewhat complicated example from the symbolization of relations in 9.5:

> Kim is a genius. So, anyone who admires Kim admires a genius.

We can show that this is valid as follows:

1. Gk $\therefore$ (x)(Axk → (∃y)(Gy • Axy))
2.