9.4: Quantifier Negation, RAA, and CP

In this section we add two equivalence rules to our system of proof and explain how to use CP and RAA within predicate logic.

The following pairs of statements are obviously equivalent:

Something is red. It is not the case that everything is not red.
Something is not red. It is not the case that everything is red.
Everything is red. It is not the case that something is not red.
Everything is not red. It is not the case that something is red.

This observation motivates the rule of quantifier negation:

Quantifier Negation (QN)
$(\exists x)\mathcal{P} \;::\; \sim(x)\sim\mathcal{P}$
$(\exists x)\sim\mathcal{P} \;::\; \sim(x)\mathcal{P}$
$(x)\mathcal{P} \;::\; \sim(\exists x)\sim\mathcal{P}$
$(x)\sim\mathcal{P} \;::\; \sim(\exists x)\mathcal{P}$

Note that these are *equivalence rules*, which means that (i) they can be applied in both directions and (ii) they can be applied to *parts* of WFFs.

Examples of applications of the QN rule

Quantifier Negation (QN)
$(\exists x)\mathcal{P} \quad :: \quad \sim(x)\sim\mathcal{P}$
$(\exists x)\sim\mathcal{P} \quad :: \quad \sim(x)\mathcal{P}$
$(x)\mathcal{P} \quad :: \quad \sim(\exists x)\sim\mathcal{P}$
$(x)\sim\mathcal{P} \quad :: \quad \sim(\exists x)\mathcal{P}$

A. 1. $\exists x(Ax \bullet Bx)$
   2. $\sim(x)\sim(Ax \bullet Bx)$ 1, QN

B. 1. $\sim(y)Cy$
   2. $(\exists y)\sim Cy$ 1, QN

C. 1. $(x)(Ax \rightarrow Ex)$
   2. $\sim(\exists x)\sim(Ax \rightarrow Ex)$ 1, QN

D. 1. $(x)\sim(\exists y)(Ax \bullet Ex)$
   2. $\sim(\exists x)(\exists y)(Ax \bullet Ex)$ 1, QN

E. 1. $(y)Ky \rightarrow (x)\sim Hx$
   2. $(y)Ky \rightarrow \sim(\exists x)Hx$ 1, QN

F. 1. $(\exists y)(By \bullet \sim(x)(Ax \vee \sim Cy))$
   2. $(\exists y)(By \bullet (\exists x)\sim(Ax \vee \sim Cy)$ 1, QN

Examples E and F here illustrate the application of QN to *parts* of WFFs.

Example Proof

1. $\sim(\mathrm{x})(\mathrm{Ax} \rightarrow \mathrm{Mx})$
2. $(\mathrm{x})(\mathrm{Rx} \rightarrow \mathrm{Mx}) \qquad \therefore (\exists \mathrm{x})(\mathrm{Ax} \cdot \sim \mathrm{Rx})$

Tip 2: When the statement in a line of a proof is the negation of quantified statement (i.e., a statement of the form $\sim(x)\mathcal{P}$ or $\sim(\exists x)\mathcal{P}$, it is *very* often useful to apply QN and instantiate using EI or UI, as the case may be.

CP, RAA and UG

CP and RAA are of course available to us in predicate logic. So far, to keep the initial statement of UG simple, we have avoided them in our examples. The use of these methods, however, requires us to add an additional restriction on UG: *One cannot universally generalize on a constant that occurs in an undischarged assumption.* The justification for this is similar the justification for the other restrictions: When we make an assumption, we may be adding special information about any of the objects we are talking about in our proof. Hence, they are no longer "arbitrary" representatives of all objects generally.

Universal Generalization (UG)
$\mathcal{P}_c$ $\therefore\ (x)\mathcal{P}$ where $\mathcal{P}_c$ is an instance of $(x)\mathcal{P}$ and (a) c does not occur in $(x)\mathcal{P}$, (b) c does not occur in any premise of the argument, (c) c does not occur in a line derived by an application of EI, and (d) c does not occur in an undischarged assumption.

Example: Fallacious proof of $(x)Rx \rightarrow (x)Bx \therefore (x)(Rx \rightarrow Bx)$

1.	$(x)Rx \rightarrow (x)Bx$	
2.	Ra	Assume
3.	$(x)Rx$	2 UG (**MISTAKE!** Violation of condition (d))
4.	$(x)Bx$	1,3 MP
5.	Ba	4 UI
6.	$Ra \rightarrow Ba$	2-5 CP
7.	$(x)(Rx \rightarrow Bx)$	6 UG

To see that this argument is invalid, consider a scheme of abbreviation for "R" and "B" on which not everything is an R and on which no R is a B. For example, let "Rx" mean "x is a rabbit" and let "Bx" mean "x is a bird". Since not everything is a rabbit, "$(x)Rx$" is false. Hence, by the truth table for the material conditional, the premise "$(x)Rx \rightarrow (x)Bx$" is true. But the conclusion "$(x)(Rx \rightarrow Bx)$" is obviously false, as it says that all rabbits are birds.

Be careful not to take the restriction on UG to be more restrictive than it is. Consider the following proofs.

Proof 1.

1. $(x)(Fx \rightarrow Gx)$
2. $(x)(Fx \rightarrow Hx)$ $\quad\quad$ $\therefore (x)(Fx \rightarrow (Gx \cdot Hx))$

The proof nicely formatted:

1.	$(x)(Fx \rightarrow Gx)$	
2.	$(x)(Fx \rightarrow Hx)$	$\therefore (x)(Fx \rightarrow (Gx \cdot Hx))$
3.	Fa	Assume
4.	$Fa \rightarrow Ga$	1 UI
5.	$Fa \rightarrow Ha$	2 UI
6.	Ga	3,4 MP
7.	Ha	3,5 MP
8.	$Ga \bullet Ha$	6,7 Conj
9.	$Fa \rightarrow (Ga \bullet Ha)$	3-8 CP
10.	$(x)(Fx \rightarrow (Gx \bullet Hx))$	9 UG

The application of UG in line 10 does not violate the new condition (d) on UG because the assumption "Fa" had been discharged at line 9 (as signified by the box)!

Proof 2.

1. $(x)(Ax \to (y)Gy)$ $\quad\quad$ $\therefore (x)Ax \to (x)Gx)$

The proof nicely formatted:

1.	$(x)(Ax \to (y)Gy)$	$\therefore (x)Ax \to (x)Gx$
2.	$(x)Ax$	Assume
3.	Aa	2 UI
4.	$Aa \to (y)Gy$	1 UI
5.	$(y)Gy$	3,4 MP
6.	Ga	5 UI
7.	$(x)Gx$	6 UG
8.	$(x)Ax \to (x)Gx$	2-7 CP

Again the application of UG in line 7 does not violate condition (d), even though the assumption in line 2 has not be discharged because the constant "a" in the formula in line 6 that is generalized upon in line 7 *does not occur in* that assumption. Once again, the lesson is not to take condition (d) in UG to be more restrictive than it is.

Proof 1 from p. 5 suggests another tip.

Tip 3: If the conclusion of an argument is a universally quantified statement of the form $(x)(\mathcal{P} \rightarrow \mathcal{Q})$, use CP to prove an instance $\mathcal{P}_c \rightarrow \mathcal{Q}_c$ and then apply UG. (Be sure your choice of individual constant c won't violate any of the conditions on UG.)

When the conclusion of an argument is a particular statement, RAA is often effective, as it gives us a temporary universal premise to work with.

Another Example

1. $(x)(Px \rightarrow Sx)$
2. $Pa \vee Pb$ $\quad\quad\quad\quad \therefore (\exists x)Sx$

The proof nicely formatted:

1.	$(x)(Px \rightarrow Sx)$	$\therefore (\exists x)Sx$
2.	$Pa \lor Pb$	
3.	$\sim(\exists x)Sx$	Assume
4.	$(x)\sim Sx$	3 QN
5.	$Pa \rightarrow Sa$	1 UI
6.	$\sim Sa$	4 UI
7.	$\sim Pa$	5,6 MT
8.	Pb	2,7 DS
9.	$Pb \rightarrow Sb$	1 UI
10.	Sb	8,9 MP
11.	$\sim Sb$	4 UI
12.	$Sb \bullet \sim Sb$	10,11 Conj
13.	$(\exists x)Sx$	3-12 RAA

This proof illustrates a further tip from our text:

> *Tip 4*: When the conclusion of an argument is a quantified statement (i.e., a statement of the form $(\mathcal{x})\mathcal{P}$ or $(\exists \mathcal{x}\mathcal{P})$, RAA is often useful.

However, I think the following is a more useful generalization of that tip:

> *Menzel Tip*: RAA is often useful!

More specifically: When you don't see any obvious way of continuing your proof working top-down, *use RAA to try to prove your conclusion!*