9.3: Constructing Proofs

In this section we extend the method of proof introduced in Chapter 8 to predicate logic by adding four new rules, two for each quantifier.

NOTE: All of the rules of statement logic will continue to apply in predicate logic. Notably, equivalence rules will still be applicable inside quantified WFFs.

Example

All moral agents are rational. So, everything is either rational or not a moral agent.

1. $(x)(Mx \rightarrow Rx)$ $\quad\therefore\ (x)(Rx \vee \sim Mx)$
2. $(x)(\sim Mx \vee Rx)$ 1, MI
3. $(x)(Rx \vee \sim Mx)$ 2, Com

Instances of Quantified WFFs

Definition: Let c be any individual constant (i.e., any of the letters a, b, ..., u). An *instance* of a quantified WFF $(x)\mathcal{P}$ or $(\exists x)\mathcal{P}$ is any WFF obtained by the following two steps:

1. Remove the initial quantifier (x) or $(\exists x)$, as the case may be.

2. In the WFF resulting from Step 1, uniformly replace all free occurrences of the variable x in $\mathcal{P}$ with occurrences of c. We signify the resulting instance by $\mathcal{P}_c$.

Example

The WFF

$$(x)(Fx \rightarrow (\exists y)(Gy \vee Hx))$$

has instances

$$Fa \rightarrow (\exists y)(Gy \vee Ha), Fb \rightarrow (\exists y)(Gy \vee Hb), ..., Fu \rightarrow (\exists y)(Gy \vee Hu)$$

The following fail to be instances of the above WFF above for various reasons:

- $(x)(Fx \rightarrow (Ga \vee Hx))$
  - The *initial* quantifier was not removed.
- $Fb \rightarrow (\exists y)(Gy \vee Hc)$
  - Free occurrences of x are not *uniformly* replaced by a single constant.
- $Fa \rightarrow (\exists y)(Ga \vee Ha)$
  - Only free occurrences of the variable bound by the initial quantifier (i.e., x) are to be replaced by occurrences of an individual constant.
- $Fz \rightarrow (\exists y)(Gy \vee Hz)$
  - z is a variable, not an individual constant.

Universal Instantiation (UI)

UI allows us to infer from an assertion about everything (in a given universe) a corresponding statement about a given individual.

Example

All politicians are egomaniacs. Jesse is a politician. Therefore, Jesse is an egomaniac.

1.	$(x)(Px \rightarrow Ex)$	
2.	Pj	$\therefore$ Ej

One might be tempted to use MP here, but Premise 1 does not have the proper form. Premise 1 is a universally quantified statement; what we need is a conditional statement in which 'Pj' is the antecedent. However, the first premise tells us that for *everything* x, if x is a politician, then x is an egomaniac. But what goes for everything goes for Jessie, in particular. Hence, in a complete system, we should be able to infer the instance $Pj \rightarrow Ej$ from the first premise and then legitimately apply MP. The rule that enables us to do exactly this is *universal instantiation*:

Universal Instantiation (UI)
$\quad (x)\mathcal{P}$
$\therefore\ \mathcal{P}_c$
where $\mathcal{P}_c$ is any instance of $(x)\mathcal{P}$.

Given UI, we can now complete the above proof:

1.	$(x)(Px \rightarrow Ex)$	
2.	Pj	$\therefore$ Ej
3.	$Pj \rightarrow Ej$	1, UI
4.	Ej	2,3 MP

Errors to Avoid

Be very sure that you only apply UI to statements that universally quantified, not to universally quantified *parts* of statements.

Example

1. $\sim(x)Ax$
2. $\sim Ac$ 1, UI **(MISTAKE!)**

To allow this inference would be to allow an argument like this: "Not everyone is an Aggie. Therefore, Lyle Lovett is not an Aggie."

Example

1. $(x)Bx \rightarrow \sim(\exists y)Oy$
2. $Bb \rightarrow \sim(\exists y)Oy$ 1, UI **(MISTAKE!)**

To allow this inference would be to allow an argument like this: "If everyone has a brother, then no one is an only child. Therefore, if George W. Bush has a brother, then no one is an only child."

Contrast the second of these examples with a correct application of UI:

1. $(x)(Bx \rightarrow \sim(\exists y)Oy)$
2. $Bb \rightarrow \sim(\exists y)Oy$ 1, UI **(OK!)**

The scope of the quantifier makes all the difference!

Existential Generalization (EG)

We can illustrate the need for our next rule with the following argument:

All politicians are egomaniacs. Jesse is a politician. Therefore, someone is an egomaniac.

We've just seen that we can infer that Jesse is an egomaniac from these same premises. So it clearly follows that *someone* is an egomaniac — Jesse, for example. What we need is a rule that enables us to infer that something in general has a given property from the fact that some particular object does; we call this rule *existential generalization*:

Existential Generalization (EG)
$\quad \mathcal{P}_c$
$\therefore \quad (\exists x)\mathcal{P}$
where $\mathcal{P}_c$ is any instance of $(\exists x)\mathcal{P}$.

Given EG, we can continue the argument above to reach our desired conclusion:

1.	$(x)(\mathrm{Px} \rightarrow \mathrm{Ex})$	
2.	Pj	$\therefore (\exists x)\mathrm{Ex}$
3.	$\mathrm{Pj} \rightarrow \mathrm{Ej}$	1, UI
4.	Ej	2,3 MP
5.	$(\exists x)\mathrm{Ex}$	4 EG

Here are some applications of EG, some correct, some incorrect:

1. Fa
2. $(\exists x)Fx$ 1 EG

- Correct!

1. Gx
2. $(\exists y)Gy$ 1 EG

- **Mistake!** LIne 1 is not an instance of line 2 (why?)

1. $Sb \vee Rc$
2. $(\exists z)(Sz \vee Rc)$ 1 EG

- Correct!

1. $Sb \vee Rc$
2. $(\exists y)(Sy \vee Ry)$ 1 EG

- **Mistake!** LIne 1 is not an instance of line 2 (why?)

1. $Ma \bullet Sa$
2. $(\exists x)(Mx \bullet Sx)$ 1 EG

- Correct!

1. $Ja \bullet {\sim}Kb$
2. $(\exists x)Jx \bullet {\sim}Kb$ 1 EG

- **Mistake!** LIne 1 is not an instance of line 2 (why?)

1. $Ma \bullet Sa$
2. $(\exists x)(Mx \bullet Sa)$ 1 EG

- Correct!

Existential Instantiation (EI)

We can illustrate the need for our next rule with the following argument:

All politicians are egomaniacs. Some politicians are ruthless. Therefore, some egomaniacs are ruthless.

The second premise here tells us that there are ruthless politicians, but it doesn't give us any names. However, assuming the premise is true, we can give an arbitrary ruthless politician a temporary name — "Pat", say — and reason about him or her. Specifically, from the first premise, we can infer that, if Pat is a politician, he or she is an egomanic and, hence, since he or she is a ruthless politician, he or she must therefore be a ruthless egomanic. But since Pat is, it follows by existential generalization that *something* is.

Note that the validity of this reasoning requires that we use a *new* name that is not already used for someone else in the context, as we might have made assumptions about that person that might not be true of an arbitrary ruthless politician. Furthermore, we don't want to end our reasoning by drawing any conclusions about *Pat*, since that is just a temporary name for an unknown ruthless politician.

The rule that permits us to introduce temporary names and reason about them in this way is called *existential instantiation*:

Existential Instantiation (EI)
$\quad (\exists x)\mathcal{P}$
$\therefore \ \mathcal{P}_c$
where $\mathcal{P}_c$ is an instance of $(\exists x)\mathcal{P}$ and (i) c does not occur earlier in the proof and (ii) c does not occur in the last line of the proof.

Restriction (i) here ensures that we are making no special assumption about the arbitrary object signified by our new name; restriction (ii) ensures that our name is truly temporary and is only used to derive more general conclusions.

Given EI, we can recreate the informal proof given for the argument above:

1.	$(x)(Px \rightarrow Ex)$	
2.	$(\exists x)(Px \bullet Rx)$	$\therefore\ (\exists x)(Ex \bullet Rx)$
3.	$Pp \bullet Rp$	2, EI
4.	$Pp \rightarrow Ep$	1, UI
5.	Pp	3 Simp
6.	Ep	4,5 MP
7.	Rp	3 Simp
8.	$Ep \bullet Rp$	6,7 Conj
9.	$(\exists x)(Ex \bullet Rx)$	8 EG

NB!! Note that we applied EI before UI. Note what happens if we'd applied UI first:

Tip 1: When you have lines with both universal and existential quantified statements, *APPLY EI BEFORE UI!*

Violating Restriction (i)

To illustrate the importance of restriction (i), consider the following "proof":

1.	$(\exists x)Dx$	
2.	$(\exists y)Cy$	$\therefore (\exists x)(Dx \bullet Cx)$
3.	Da	1, EI (OK)
4.	Ca	2, EI **(MISTAKE!)**
5.	$Da \bullet Ca$	3,4, Conj
6.	$(\exists x)(Dx \bullet Cx)$	5 EG

Letting "Dx" mean "x is a dog" and "Cx" mean "x is a cat", from the premises that there are dogs and there are cats we have proved that something is both a dog and a cat!

Universal Generalization (UG)

We can illustrate the need for our next rule with the following argument:

All politicians are ambitious. All ambitious people are high strung. Therefore, all politicians are high strung.

To prove this we use UI to generate instances of the premises, and infer an instance of the conclusion on which we generalize. The rule we need to allow this is *universal generalization*:

Universal Generalization (UG)
$\quad \mathcal{P}_c$ $\therefore \quad (x)\mathcal{P}$ where $\mathcal{P}_c$ is an instance of $(x)\mathcal{P}$, and (i) c does not occur in a premise of the argument, (ii) c does not occur in a previous line derived by an application of EI, and (iii) c does not occur in $(x)\mathcal{P}$.

Here is a proof of our argument that demonstrates UG:

1. $(x)(Px \rightarrow Ax)$
2. $(y)(Ay \rightarrow Hy)$ $\therefore (x)(Px \rightarrow Hx)$
3. $Pa \rightarrow Aa$ 1, UI
4. $Aa \rightarrow Ha$ 2, UI
5. $Pa \rightarrow Ha$ 3,4, HS
6. $(x)(Px \rightarrow Hx)$ 5 UG

Errors to Avoid

There are three restrictions on the application of UG; consequently, there are three common errors to avoid in its application, one for each restriction. These errors are illustrated in the following examples.

Violating restriction (i): Generalizing on a constant that occurs in a premise

1. Ac
2. $(x)Ax$ 1, UG **(MISTAKE!)**

This would be like arguing as follows: "Socrates is snub-nosed. Therefore, everyone is snubnosed."

Violating restriction (ii): Generalizing on a constant occurring in a line derived by EI

1. $(\exists x)Ax$
2. Ac 1, EI
3. $(x)Ax$ 2, UG **(MISTAKE!)**

This would be like arguing as follows: "There are tall people. So let Pat be an example of one. Therefore, everyone is tall."

Violating restriction (iii): Generalizing on a constant that occurs in the generalization

1. $(x)(Ax \rightarrow Bx)$
2. $Ac \rightarrow Bc$ 1, UI
3. $(x)(Ac \rightarrow Bx)$ 2, UG **(MISTAKE!)**

This would be like arguing as follows: "Everyone who has a brother has a sibling. Hence, if George, in particular, has a brother, then George has a sibling. Therefore, if George has a brother, then everyone has a sibling."

Example Proof

1. $(x)(\sim Px \rightarrow Qx)$
2. $(\exists x)(Rx \bullet \sim Qx)$ $\therefore$ $(\exists y)(Ry \bullet Py)$
3.

A complete proof of the above argument, nicely formatted:

1.	$(x)(\sim Px \rightarrow Qx)$	
2.	$(\exists x)(Rx \bullet \sim Qx)$	$\therefore (\exists y)(Ry \bullet Py)$
3.	$Ra \bullet \sim Qa$	2, EI
4.	$\sim Pa \rightarrow Qa$	1, UI
5.	$\sim Qa$	3, Simp
6.	$\sim\sim Pa$	4,5 MT
7.	Pa	6 DN
8.	Ra	3, Simp
9.	$(Ra \bullet Pa)$	7,8 Conj
10.	$(\exists y)(Ry \bullet Py)$	9 EG