9.1: Predicates and Quantifiers

Expressive Limitations of Statement Logic

Many intuitively valid arguments in ordinary language cannot be represented as valid in statement logic. For example:

- This house is red. Therefore, something is red.
- All logicians are exceptional. Saul is a logician. Therefore, Saul is exceptional.
- No politicians are honest. Some politicians are administrators. Therefore, some administrators are not honest.
- Every horse is an animal. Therefore, every head of a horse is a head of an animal.

Represented in statement logic (under appropriate schemes of abbreviation), these arguments look like this:

- $H \therefore R$
- $M. S \therefore T$
- $P. A \therefore \sim H$
- $H \therefore A$

Obviously, all three of these symbolized arguments are invalid in statement logic. (You should know how to prove that!)

The problem: The validity of the above arguments rests in large measure on "subsentential" components of the constituent statements — e.g., the name *Saul*, the plural common noun *logicians*, the verb phrase *is exceptional*, and the "quantifier" *All*. Because the basic unit of statement logic is the atomic statement, statement logic is incapable of representing these features of the arguments that are crucial to their validity.

In this chapter, we will extend statement logic with the apparatus and methods necessary for representing the above arguments adequately and proving their validity.

Individual Constants, Predicates, and Quantifiers

Recall the four Standard Forms of categorical statements:

Categorical Statement	**Form**
Universal affirmative	All S are P.
Universal negative	No S are P.
Particular affirmative	Some S are P.
Particular negative	Some S are not P.

To represent these statements, we need to supplement the language of statement logic with new elements corresponding to those noted above. The first kinds of elements we need are those corresponding to names and verb phrases.

Individual constants: $a, ..., u$

Predicate letters: $A, ..., Z$

Individual constants will represent names of individual things like the logician Saul Kripke, the city of Austin, and the number 17. Predicate letters standing alone are just statement letters. However, when combined with individual constants and names they stand for verb phrases like 'is a man' and 'is mortal', and relational predicates like 'is the head of' (which we will not study in this section). Individual constants and predicate letters will enable to represent the subsentential structure simple atomic sentences and boolean combinations of such.

Example:

Let 'L' represent 'is a logician' and let 's' represent 'Saul' and 'a' represent 'Arthur'.

- Saul is a logician: Ls

- Saul and Arthur are logicians: $\mathrm{Ls} \bullet \mathrm{La}$

To represent more complex sentences like categorical statements, we need to introduce the apparatus of quantifiers and variables.

Individual variables: $\mathrm{v}, \mathrm{w}, \mathrm{x}, \mathrm{y}, \mathrm{z}$

An individual variable can be thought of as a placeholder for a name in a sentence.

Example:

- _____ is a logician: Lx (or Lw, or Ly, etc.)

- _____ is the head of ..... : Hxy (or Hwv, or Hvy, or Hyx, etc.)

Expressions like 'Lx' and 'Hxy' that contain a variable where a name could go are called *statement functions* (because replacing a variable with a constant yields a statement).

Statement functions are used together with *quantifiers* to symbolize categorical statements and other sentences containing 'all', 'every', 'no', 'some', 'at least one', etc.

Quantifiers: (x), $(\exists \mathrm{x})$, (y), $(\exists \mathrm{y})$, and so on, for the other variables

'(x)', '(y)', etc are called *universal quantifiers*, and are used to translate universal categorical statements and, more generally, statements containing 'all', 'every', 'each', and (usually) 'any'.

'$(\exists x)$', '$(\exists y)$' etc are called *existential quantifiers*, and are used to translate particular categorical statements and, more generally, statements containing 'some', 'at least one' (as in "At least one student scored a 100"), 'a' (as in "A woman won the race"), 'there is' (as in "There is a planet beyond Pluto"), and 'there are' (as in "There are cats that have no fur").

As we will see, 'No' (as in "No politicians are honest") can translated equally well by either quantifier.

Translating Categorical Statements

Universal Affirmatives

Let 'L' stand for 'is a logician' again and let 'E' stand for 'is exceptional'.

- Ordinary English: "All logicians are exceptional."
- Can be paraphrased as: "Everything is such that, if it is logician, then it is exceptional."
- In logical English ("Logicese"): "For all individuals x, if x is a logician, then x is exceptional."
- Fully translated: $(x)(Lx \rightarrow Ex)$

In general, "All S are P" is translated into predicate logic as $(x)(Sx \rightarrow Px)$, where x is any variable.

Note that *the arrow usually goes with the universal quantifier*.

Universal Negatives

Let ‘P’ stand for ‘is a politician’ and let ‘H’ stand for ‘is honest’.

- Ordinary English: ‘No politicians are honest’ .

- Can be paraphrased as:
  - Everything is such that, if it is politician, then it is not honest.
  - It is false that something is both a politician and honest.

- In logical English:
  - For all individuals x, if x is a politician, then x is not honest.
  - It is not the case that there is an individual x such that x is a politician and x is a honest.

- Fully translated:
  - $(\mathrm{x})(\mathrm{Px} \rightarrow \sim \mathrm{Hx})$
  - $\sim(\exists \mathrm{x})(\mathrm{Px} \bullet \mathrm{Hx})$

In general, “No S are P” is translated into predicate logic as either $(x)(Sx \rightarrow \sim Px)$ or $\sim(\exists x)(Sx \bullet Px)$, where x is any variable.

Particular Affirmatives

Let 'P' stand for 'is a politician' again and let 'A' stand for 'is an administrator'.

- Ordinary English: "Some politicians are administrators."
- Can be paraphrased as: "Something is such that it is both a politician and an administrator."
- In logical English: "For some individual x, x is a politician and x is an administrator."
- Fully translated: $(\exists \mathrm{x})(\mathrm{Px} \bullet \mathrm{Ax})$

In general, "Some S are P" is translated into predicate logic as $(\exists x)(Sx \bullet Px)$, where x is any variable.

Note that *the dot usually goes with the existential quantifier*.

Particular Negatives

- Ordinary English: "Some administrators are not honest."
- Can be paraphrased as: "Something is such that it is both an administrator and not honest."
- In logical English: "For some individual x, x is an administrator and x is not honest."
- Fully translated: $(\exists \mathrm{x})(\mathrm{Ax} \bullet \sim \mathrm{Hx})$

In general, "Some S are not P" is translated into predicate logic as $(\exists x)(Sx \bullet \sim Px)$, where x is any variable.

Stylistic Variants for Categorical Statements

Recall that each type of categorical statement has both a standard form and multiple *stylistic variants* that, essentially, mean the same thing.

Universal affirmative: $(x)(Sx \rightarrow Px)$	Universal negative: $(x)(Sx \rightarrow \sim Px)$ $\sim(\exists x)(Sx \bullet Px)$
All men are animals.	*No politicians are logicians.*
Every man is an animal.	No politician is a logician.
Each man is an animal.	All politicians are nonlogicians.
Men are animals.	No one is a politician if a logician.
Any man is an animal.	There are no politicians who are logicians.
Anything that is a man is an animal.	All politicians fail to be logicians.
Only animals are men.	Only nonlogicians are politicians.

Particular affirmative: $(\exists x)(Sx \bullet Px)$	Particular negative: $(\exists x)(Sx \bullet \sim Px)$
Some women are actors.	*Some cyclists are not triathletes.*
Some woman is an actor.	Some cyclist is not a triathlete.
At least one woman is an actor.	At least one cyclist isn't a triathlete.
There is a woman who acts.	There is a nontriathlete cyclist.
Something is both a woman and an actor.	Not every cyclist is a triathlete.

Note that "Not every S is a P" is more faithfully translated as $\sim(x)(Sx \rightarrow Px)$. As we will prove, this is equivalent to $\exists x(Sx \bullet \sim Px)$.

Although they are the most common forms containing quantifiers, not every quantified statement is a categorical statement or a stylistic variant of one. Notably, we often make unqualified quantified statements, that is, statements where we are talking about everything or something in general.

There are dogs.	$(\exists x)Dx$
A god exists.	$(\exists x)Gx$
Some things are best left unsaid.	$(\exists x)Bx$
Everything is beautiful.	$(x)Bx$
All things are known by God.	$(x)Kxg$

And, of course, it is possible to form boolean combinations of quantified statements.

There are dogs and cats.	$(\exists x)Dx \bullet (\exists y)Cy$
No gods exist.	$\sim(\exists x)Gx$ or $(x)\sim Gx$
If no gods exists, then the universe has no purpose.	$\sim(\exists x)Gx \rightarrow \sim U$

A Grammar for Predicate Logic

Definition: If P is a predicate letter and $t_1, ..., t_n$ are individual constants or variables, then $Pt_1...t_n$ is an *atomic formula*. (Note that the case where $n = 0$ tells us that our statement letters from statement logic are also atomic formulas.)

Let $\mathcal{P}$ and $\mathcal{Q}$ stand for any expression (i.e., any string of symbols) in the language of predicate logic; then we define the notion of a *well-formed formula* (*WFF*) of predicate logic as follows:

1. Every atomic formula is a WFF (of predicate logic).

2. If $\mathcal{P}$ is a WFF, so is $\sim\mathcal{P}$.

3. If $\mathcal{P}$ and $\mathcal{Q}$ are WFFs, so are $(\mathcal{P} \bullet \mathcal{Q})$, $(\mathcal{P} \vee \mathcal{Q})$, $(\mathcal{P} \rightarrow \mathcal{Q})$, and $(\mathcal{P} \leftrightarrow \mathcal{Q})$.

4. If $\mathcal{P}$ is a WFF and x any variable, then $(x)\mathcal{P}$ and $(\exists x)\mathcal{P}$ are WFFs.

Nothing else is a WFF of predicate logic.

Quantifier Scope, Bound and Free Variables, Instances

Definition: The *scope* of a quantifier in a formula $\mathcal{P}$ is the shortest WFF occurring immediately to the right of the quantifier.

Examples

In the WFF

$$(x)Fx \,\&\, Pa$$

the scope of the quantifier (x) is Fx.

In the WFF

$$(x)[(\exists y)Fxy \,\&\, (\exists z)(Gyz \leftrightarrow Hzx)]$$

the scope of the quantifier (x) is $[(\exists y)Fxy \,\&\, (\exists z)(Gyz \leftrightarrow Hzx)]$, that of $(\exists y)$ is Fxy, and that of $(\exists z)$ is $(Gyz \leftrightarrow Hzx)$.

Definition: An occurrence of a variable x that is in the scope of a quantifier for that variable (i.e., a quantifier of the form (x) or $(\exists x)$) is said to be *bound*. An occurrence of a variable that is not bound is said to be *free*.

Examples

In the WFF

$$(x)Fx \;\rightarrow\; Qx$$

the first occurrence of the variable x (not counting its occurrence in the quantifier) is bound in the WFF $(x)Px \rightarrow Qx$; the second occurrence is free.

In the WFF

$$(x)[(\exists y)Fxy \,\&\, (\exists z)(Gyz \leftrightarrow Hux)]$$

all occurrences of x are bound; the occurrence of y in Fxy is bound but its occurrence in Gyz is free; the occurrences of z in Gyz is bound; and the occurrence u in Hux is free.