8.2: Five Equivalence Rules

Comment: Recall that two statements are logically equivalent just in case they are true on exactly the same truth value assigments. Logically equivalent statements thus express the same information. Hence, given a statement p, one can always validly infer a logically equivalent statement q. This warrants the formation of rules of inference — known as *equivalence rules* — based upon logical equivalence. They provide us with explicit patterns of logical equivalence that we can use to infer new statements from given statements in a proof.

Because logically equivalent statements express the same information, one can always replace any statement-part p of any statement q with a statement p' that is logically equivalent to p and the resulting statement q' will be logically equivalent to q; it will express the same information. Consequently, equivalence rules apply not only to the entire statement in a line of a proof, but to *statement-parts* of a statement in a line of a proof. This is the main difference between implicational rules and equivalence rules.

All of our equivalence rules are of the form $p :: q$, where the four-dot symbol (: :) indicates that p is logically equivalent to q. Such a rule tells is that, in the context of a proof, we may replace any occurrence of p in a line of a proof (even if p is only part of a statement on a line) with q (or any occurrence of q with p) and validly infer the result.

We illustrate with the first of our equivalence rules:

Rule 9: Double negation (DN) $p :: \sim\sim p$

Rule 9 thus tells us that if p is a statement in a proof, or a part of a statement in a proof, we can replace it with $\sim\sim p$ and add the resulting statement to the proof as a further step, justified by DN. We can also go from "right to left": that is, if $\sim\sim p$ is a (part of a) statement in a proof, we can replace it with p and add the resulting statement to the proof as a further step, justified by DN.

Example: A proof for $\sim F \rightarrow \sim R$, $R \therefore F$

1. $\sim F \rightarrow \sim R$
2. $R \qquad \therefore\ F$

Rule 10: Commutation (Com) $p \vee q :: q \vee p$

$p \bullet q :: q \bullet p$

Example: A proof for $P \rightarrow \sim (B \bullet O)$, $O \bullet B \therefore \sim P$

1. $P \rightarrow \sim (B \bullet O)$
2. $O \bullet B \qquad \therefore\ \sim P$

Rule 11: Association (As)	$p \vee (q \vee r) :: (p \vee q) \vee r$
	$p \bullet (q \bullet r) :: (p \bullet q) \bullet r$

Example: A proof for $(C \vee R) \vee D$, $\sim (R \vee D) \therefore C$

1. $(C \vee R) \vee D$
2. $\sim (R \vee D)$ $\quad \therefore C$

Rule 12: DeMorgan's laws (DeM)	$\sim (p \vee q) :: \sim p \bullet \sim q$
	$\sim (p \bullet q) :: \sim p \vee \sim q$

Example: A proof for $(E \bullet D) \vee (\sim E \bullet \sim D)$, $\sim E \therefore \sim D$

1. $(E \bullet D) \vee (\sim E \bullet \sim D)$
2. $\sim E$ $\quad \therefore \sim D$

Rule 13: Contraposition (Cont) $p \rightarrow q :: \sim q \rightarrow \sim p$

Example: A proof for $(W \rightarrow D) \rightarrow \sim C$, $\sim D \rightarrow \sim W \therefore \sim C$

1. $(W \rightarrow D) \rightarrow \sim C$
2. $\sim D \rightarrow \sim W \qquad \therefore \sim C$

Tip 6: It is often useful to consider logically equivalent forms of the conclusion.

Example

1. $\sim G \rightarrow \sim A$
2. $\sim H \rightarrow \sim B$
3. $\sim (G \bullet H) \qquad \therefore \sim (A \bullet B)$
4.

$\sim (A \bullet B)$

> *Tip 7*: Both conjunction and disjunction can lead to useful applications of De Morgan's laws.

That is, when you see a conjunction or disjunction, or a negated conjunction or a negated disjunction, scan your premises and derived lines to see if a transformation using DeM might be useful.

Example

1. $\sim J \vee \sim L$
2. $\sim (J \bullet L) \rightarrow \sim M$
3. $\sim E \vee (M \vee \sim S) \quad \therefore \sim (S \bullet E)$

$\sim (S \bullet E)$