

# Chapter 8

## *Statement Logic: Proofs*

### Introduction

- Truth tables provide a natural representation of the *meanings* of our five logical operators.
- This in turn provided us with a notion of *validity* for statement logic.
- Truth tables provided us with an effective *method* for determining validity.

*BUT:* The truth table method can be *cumbersome!*

- The focus of the rest of the course will be on systems of *natural deduction*
- In natural deduction, one draws from a set of inference rules to demonstrate, step by step, that the conclusion of an argument follows from the premises.

### *Advantages of Natural Deduction*

- Generally less cumbersome than truth tables
- Reflects very clearly our intuitive patterns of valid reasoning

## 8.1 — Implicational Rules of Inference

A *proof* is a series of “steps” that leads, by way of valid inference rules, from the premises of a symbolic argument to its conclusion.

- An inference rule is, basically, a *permission slip*.
- An inference rule tells us that, whenever we have statements  $p_1, p_2, \dots, p_n$  of certain forms, we are *permitted* to infer a statement  $q$  of a certain form.
- We will express inference rules as:  $p_1, p_2, \dots, p_n \therefore q$ 
  - You can read this as: “ $q$  follows from  $p_1, p_2, \dots, p_n$ ”
- An inference rule  $p_1, p_2, \dots, p_n \therefore q$  is *valid* if, whenever  $p_1, p_2, \dots, p_n$  are true,  $q$  must be true as well.

Thus, because all of the “steps” in a proof are permitted by valid inference rules:

*If the premises of a proof are true, the conclusion must be true as well.*

And that is just what we would hope would be true of any reasonable notion of a proof. Importantly however:

- *Inference rules are purely syntactic, purely “mechanical”!*
  - Although valid, they are in and of themselves simply rules that permit us to write down one statement given certain others.

Our first five rules formalize the famous valid argument forms we examined in Section 1.2:

**Rule 1.** *Modus Ponens (MP)*

$$\begin{array}{l} p \rightarrow q \\ p \\ \therefore q \end{array}$$

**Rule 2.** *Modus Tollens (MT)*

$$\begin{array}{l} p \rightarrow q \\ \sim q \\ \therefore \sim p \end{array}$$

**Rule 3.** *Hypothetical Syllogism (HS)*

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \therefore p \rightarrow r \end{array}$$

**Rule 4.** *Disjunctive Syllogism (DS)*

$$\begin{array}{ll} p \vee q & p \vee q \\ \sim p & \sim q \\ \therefore q & \therefore p \end{array}$$

**Rule 5.** *Constructive Dilemma (CD)*

$$\begin{array}{l} p \vee q \\ p \rightarrow r \\ q \rightarrow s \\ \therefore r \vee s \end{array}$$

To these we add three additional rules. First, *simplification*, which permits us to infer either conjunct of a conjunction:

**Rule 6.** *Simplification (Simp)*

$$\begin{array}{ll} p \cdot q & p \cdot q \\ \therefore p & \therefore q \end{array}$$

This rule is obviously valid, as a conjunction is true if and only if both its conjuncts are. Here is a simple example:

1. Both Pierre Curie and Marie Curie were physicists. Therefore, Marie Curie was a physicist.

The next rule is sort of the converse of the preceding rule, as it permits us to infer a conjunction from its conjuncts:

**Rule 7.** *Conjunction (Conj)*

$$\begin{array}{l} p \\ q \\ \therefore p \bullet q \end{array}$$

Conjunction is valid for the same reason that Simplification is valid. To illustrate the rule:

2. Thomas Aquinas died in 1274. William Ockham died in 1349.  
Hence, Thomas Aquinas died in 1274 *and* William Ockham died in 1349.

Finally, we have the rule of Addition, a rule for  $\vee$ :

**Rule 8.** *Addition (Add)*

$$\begin{array}{l} p \\ \therefore p \vee q \end{array} \quad \begin{array}{l} p \\ \therefore q \vee p \end{array}$$

To illustrate the rule:

3. Thomas Paine wrote *Common Sense*. Thus, either Thomas Paine wrote common sense or Patrick Henry did.

Simplification, Conjunction, and Addition (and some of the other rules) may seem ridiculously obvious, but note:

*The point of natural deduction is to characterize the meaning of the logical operators in terms of the **inferences** we are permitted to make with them.*

Hence, we need to make these inferences *explicit*. (Note the inference pattern in Simp fails if we change ‘ $\bullet$ ’ to, e.g., ‘ $\vee$ ’.)

Note that the “metavariables”  $p$ ,  $q$ ,  $r$ , and  $s$  in a rule can be replaced by *any WFF* so long as the replacement is *uniform* throughout the *rule*.

The following are all instances of the rule MP:

$$\begin{array}{lll} A \rightarrow B & (F \cdot G) \rightarrow H & (P \rightarrow (Q \cdot R)) \rightarrow \sim(S \vee T) \\ A & (F \cdot G) & P \rightarrow (Q \cdot R) \\ \therefore B & \therefore H & \therefore \sim(S \vee T) \end{array}$$

The following are all instances of the rule MT:

$$\begin{array}{lll} A \rightarrow B & (F \cdot G) \rightarrow H & (P \rightarrow (Q \cdot R)) \rightarrow \sim(S \vee T) \\ \sim B & \sim H & \sim\sim(S \vee T) \\ \therefore \sim A & \therefore \sim(F \cdot G) & \therefore \sim(P \rightarrow (Q \cdot R)) \end{array}$$

The following is NOT an instance of MT (why not?):

$$\begin{array}{l} (F \cdot G) \rightarrow \sim H \\ H \\ \therefore \sim(F \cdot G) \end{array}$$

NB: The previous example is a valid *pattern* but it is not one of our rules!

*To be a legitimate application of a rule, the pattern exhibited in the rule must be matched **exactly**.*

## Two Simple Examples

A proof for  $W \rightarrow M, N \rightarrow (\sim M \bullet P), N \therefore \sim W$

- |    |                                    |                     |
|----|------------------------------------|---------------------|
| 1. | $W \rightarrow M$                  |                     |
| 2. | $N \rightarrow (\sim M \bullet P)$ |                     |
| 3. | $N$                                | $\therefore \sim W$ |
| 4. | $\sim M \bullet P$                 | 2,3 MP              |
| 5. | $\sim M$                           | 4, Simp             |
| 6. | $\sim W$                           | 1,5 MT              |

A proof for  $(A \vee G) \rightarrow K, K \rightarrow (B \vee F), A \bullet \sim B \therefore F$  (*in class*)

- |    |                            |                |
|----|----------------------------|----------------|
| 1. | $(A \vee G) \rightarrow K$ |                |
| 2. | $K \rightarrow (B \vee F)$ |                |
| 3. | $A \bullet \sim B$         | $\therefore F$ |
| 4. | $A$                        | 3 Simp         |
| 5. |                            |                |
| 6. |                            |                |
| 7. |                            |                |
| 8. |                            |                |
| 9. |                            |                |

## Some Tips for Constructing Proofs

- Tip 1. Always, *always*, immediately check that you copied the argument correctly.
- Tip 2. Scan the premises to see whether they fit any rule patterns.
- Tip 3. Try to find the conclusion (or elements thereof) in the premises.
- Tip 4. Apply the inference rules to break down the premises.

To illustrate:

1.  $A \rightarrow (B \rightarrow (C \vee D))$
2.  $B \cdot A$
3.  $\sim D$   $\therefore C$
- 4.
- 5.
- 6.
- 7.
- 8.

But consider another example:

1.  $\sim E \vee \sim F$
2.  $\sim E \rightarrow G$
3.  $\sim F \rightarrow H$
4.  $(G \vee H) \rightarrow J \qquad \therefore J \vee K$

- Drawing on Tip 3, we see ‘J’ occurring in the consequent of line 4.
- We know, therefore, by MP, that if we can derive ‘G v H’ we will be able to derive ‘J’.
- So we ask: Can we derive ‘G v H’?
- Answer: Yes! 1, 2, and 3 together match the pattern for CD. So, putting these together, we have:

1.  $\sim E \vee \sim F$
2.  $\sim E \rightarrow G$
3.  $\sim F \rightarrow H$
4.  $(G \vee H) \rightarrow J \qquad \therefore J \vee K$
- 5.
- 6.
- 7.

But now what? How do we derive the conclusion ‘J v K’ when ‘K’ occurs nowhere in the premises?

**Tip 5.** If the conclusion contains a statement letter that does not appear in the premises, use the rule of Addition.



One more example:

1.  $(\sim N \bullet M) \rightarrow T$
2.  $\sim O \rightarrow M$
3.  $\sim O \bullet \sim N \quad \therefore T \vee S$
- 4.

T

$T \vee S$

Add

- Note we are “working backwards” here.
- We see ‘T’, but not ‘S’, in the conclusion.
- So, drawing on Tip 5, we are assuming we will be able to prove ‘T’ alone and add ‘S’ to get the conclusion.
- We then think about what we need to prove ‘T’.
- Observing that it is the consequence of line 1, we know that ‘T’ will follow by MP if we can prove ‘ $\sim N \bullet M$ ’.
- And we know we can prove that if we can prove each conjunct.
- We can prove ‘ $\sim N$ ’ from 3 by Simp. And we can prove ‘M’ if we can prove ‘ $\sim O$ ’. But we can! From line 3, by Simp once again.