7.4 Abbreviated Truth Tables

The full truth table method of Section 7.3 is extremely cumbersome. For example, an argument with only four statement letters requires a truth table with $2^4 = 32$ rows. One with five requires a truth table with $2^5 = 64$ rows. Obviously, truth tables of these sizes are simply impractical to construct. *Abbreviated truth tables* provide a much more efficient method for determining validity.

The Abbreviated Truth Table Method

The key insight behind the method

> If we can construct just *one row* of a truth table for an argument that makes the premises true and the conclusion false, then we will have shown the argument to be invalid. If we *fail* at such an attempt, we will have shown the argument to be valid.

The Method Applied to an Invalid Argument

Recall the symbolized argument from the lecture for §7.3:

$$\mathrm{A} \rightarrow (\sim\mathrm{B} \vee \sim\mathrm{W}), \mathrm{W} \therefore \sim\mathrm{A}$$

As we showed using the full truth table method, the argument is invalid. We will apply the abbreviated method to derive the same result, albeit in a single line.

1. *Write down* the symbolized argument:

A	→	(~B	∨	~	W),	W	∴	~	A

2. *Assume* that the premises are true and the conclusion false — we thus *challenge* the argument to prove to us that its valid!

A	→	(~B	∨	~	W),	W	∴	~	A
	T					T		F	

3. *Copy* the truth value assigned to W (and, in general, to any statement letter) to its other occurrences:

   - As before, we will set newly added truth values in *red* and we will highlight the truth values that were used to justify their addition in *yellow*.

A	→	(~B	∨	~	W),	W	∴	~	A
	T				T	T		F	

4. *Calculate* the truth values of compound (sub)formulas whenever you know the truth values of (enough of) their component parts.

   - Thus, we can calculate that ~W is false in virtue of our assumption that W is true:

A	→	(~B	∨	~	W),	W	∴	~	A
	T			F	T	T		F	

- And we can calculate that A must be true given that ~A is false:

A	→	(~	B	∨	~	W),	W	∴	~	A
	T				F	T	T		F	T

- Having calculated A's truth value, we copy it over to its other occurrence:

A	→	(~	B	∨	~	W),	W	∴	~	A
T	T				F	T	T		F	T

- Now, note that this gives us enough information to calculate the truth value of the consequent (~B ∨ ~W). For:
  - We have *assumed at the outset* that our premise A → (~B ∨ ~W) is true.
  - And we have *calculated* that the antecedent A is true.
  - By the truth table for conditionals $p \rightarrow q$, in order for a conditional with a true antecedent to be true, the *consequent* must *also* be true.
  - Hence, (~B ∨ ~W) must be true!

A	→	(~	B	∨	~	W),	W	∴	~	A
T	T			T	F	T	T		F	T

- But now that we have deduced that (~B ∨ ~W) must be true (given our initial assumption), we know by the truth table for disjunctions $p \vee q$ that, because its right disjunct ~W is false, its left disjunct ~B must be true — for a disjunction is true if and only if at least one of its disjuncts is. Thus:

A	→	(~	B	∨	~	W),	W	∴	~	A
T	T	T		T	F	T	T		F	T

- And this, of course, enables us now to deduce that B is false, which completes the row:

A	→	(~B	∨	~W),	W	∴	~A
T	T	T F	T	F T	T		F T

- So we have identified a row that makes the premises of our argument true and the conclusion false, so we have thereby demonstrated that the argument is invalid when A is true, B is false, and W is true.
- So we complete the abbreviated truth table by recording this invalidating truth value assignment into the table.

A	B	W	A	→	(~B	∨	~W),	W	∴	~A
T	F	T	T	T	T F	T	F T	T		F T

- Much shorter than the full truth table! To remind you:

A	B	W	A	→	(~B	∨	~W),	W	∴	~A
T	T	T		F	F	F	F	T		F
T	T	F		T	F	T	T	F		F
T	F	T		T	T	T	F	T		F
T	F	F		T	T	T	T	F		F
F	T	T		T	F	F	F	T		T
F	T	F		T	F	T	T	F		T
F	F	T		T	T	T	F	T		T
F	F	F		T	T	T	T	F		T

- Note that row 3 is exactly the row that we just constructed using the abbreviated truth table method.

The method applied to a valid argument

What happens if the argument in question is *valid*? We demonstrate with a further example. We'll cut right to the chase with a symbolized argument without worrying about the English argument it symbolizes.

We begin with the usual hypothesis that the premises are true and the conclusion false:

W	∨	J,	(W	→	Z)	∨	(J	→	Z),	~Z	∴	~	(W	•	J)
	T					T				T		F			

Since the conclusion ~(W • J) is false, its immediate component (W • J) must be true:

W	∨	J,	(W	→	Z)	∨	(J	→	Z),	~Z	∴	~	(W	•	J)
	T					T				T		F		T	

From the truth table schema for conjunctions $p \bullet q$, the only way for (W • J) to be true is if both W and J are true. So we record this information beneath the two statement letters:

W	∨	J,	(W	→	Z)	∨	(J	→	Z),	~Z	∴	~	(W	•	J)
	T					T				T		F	T	T	T

And having calculated the truth values for W and J, we copy them over to their other occurrences in the table:

W	∨	J,	(W	→	Z)	∨	(J	→	Z),	~Z	∴	~	(W	•	J)
T	T	T	T			T	T			T		F	T	T	T

So far so good — by the truth table for disjunctions, all we needed was for at least one of W and J to be true in order for W ∨ J to be true and, hence, to remain consistent with our initial assumption that the premises are true. So let us continue.

From our assumption that ~Z is true, we calculate that Z is false:

W	∨	J,	(W	→	Z)	∨	(J	→	Z),	~	Z	∴	~(	W	•	J)
T	T	T	T			T	T			T	F		F	T	T	T

And, accordingly, we copy that truth value over to the other two occurrences of Z:

W	∨	J,	(W	→	Z)	∨	(J	→	Z),	~	Z	∴	~(	W	•	J)
T	T	T	T		F	T	T		F	T	F		F	T	T	T

Calculating truth values for (W → Z) and (J → Z) we have:

W	∨	J,	(W	→	Z)	∨	(J	→	Z),	~	Z	∴	~(	W	•	J)
T	T	T	T	F	F	T	T	F	F	T	F		F	T	T	T

But now there is a problem!

- On the basis of our original assumption that the premises are true, we have calculated that both disjuncts of our second premise (W → Z) ∨ (J → Z) are false.
- But, by the truth table for disjunctions $p \vee q$, if both disjuncts of a disjunction are false, the disjunction itself is false.
- Hence, we have to calculate that (W → Z) ∨ (J → Z) is *false*.
- But *this contradicts our original assumption* that all of the premises — hence, in particular, (W → Z) ∨ (J → Z) — are true!
- So our attempt to make the premises true and the conclusion false *failed*!
- And this shows that the argument is *valid*. *(continued ↪)*

- We indicate this failure with a slash marking the spot where the conflict arose:

W	∨	J,	(W	→	Z)	∨	(J	→	Z),	~	Z	∴	~	(W	•	J)
T	T	T	T	F	F	/	T	F	F	T	F		F	T	T	T