7.2 Truth Tables

Truth tables provide a simple method for testing whether or not a given argument is valid. They can also be used to test for certain logical properties of statements.

Definition (sorta): A compound statement is *truth functional* if its truth value (i.e., its truth or falsity) is completely determined by the truth values of its component statements.

- *The compound statements of statement logic are all truth functional!* We will be able to calculate their truth values given only the truth values of their atomic parts.

Negations

SEMANTIC RULE FOR NEGATIONS
The truth value of a negation $\sim p$ is the opposite of the truth value of its immediate component p.

This rule is captured in the following *truth table schema*:

Truth table schema for negations[1]

p	$\sim p$
T	F
F	T

[1] This is a truth table *schema* because it is really a *pattern* that characterizes infinitely many truth tables at once — one for every possible formula p.

Conjunctions

SEMANTIC RULE FOR CONJUNCTIONS
A conjunction is true if both its conjuncts are true; otherwise it is false.

Truth table schema for conjunctions

p	q	$p \bullet q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunctions

SEMANTIC RULE FOR DISJUNCTIONS
A disjunction is false if both its disjuncts are false; otherwise it is true.

Truth table schema for disjunctions

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

As the truth table makes explicit, we understand "V" to express *inclusive* rather than *exclusive* disjunction.

- That is: we allow a disjunction to be true if *at least one* of the disjuncts is true.

Exclusive disjunction requires that *exactly one* of the disjuncts be true in order for the disjunction to be true. (If this were the meaning we desired for "V", how would we alter the truth table above?)

The choice of the inclusive meaning is reasonable, as

1) many instances of disjunction in English are inclusive, and
2) the exclusive meaning can be expressed very naturally in terms of the inclusive and negation.

On the first point consider a sentence like the following, which might appear in a job advertisement:

> *Applicants must have a BS in computer science* ***or*** *at least 3 years of programming experience.*

Clearly, a company would not discard an application from someone with both the degree and the experience.

On the second point, note that an exclusive disjunction can be expressed simply by expressing an inclusive disjunction and then explicitly ruling out the possibility of both disjuncts being true. Letting "⊗" express exclusive disjunction:

$$(p \otimes q) \equiv_{df} (p \vee q) \bullet (\sim p \vee \sim q)$$ [2]

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[2] "$\equiv_{df}$" here abbreviates "means by definition".

Material Conditionals

SEMANTIC RULE FOR CONDITIONALS
A conditional is false if its antecedent is true and its consequent is false; otherwise it is true.

Truth table schema for conditionals

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Natural language conditionals — statements of the form "if ... then ..." — are in fact very complicated. The truth values of some, in particular, are not completely determined by the truth values of their component statements. One semantic fact, however, ties all conditionals together, namely, that they are false if their antecedent is true and their consequent is false. The material conditional is distinguished by the fact that it is false *only* under those conditions and true under all others.

The material conditional is not entirely a logicians' invention; there many instances of it in natural language. For example, suppose I were to tell you:

> *If you get an A on the final exam, you will get an A for the course.*

The only case in which you would be able to accuse me of lying to you (hence, of having said something false) would be the case where in fact you get an A on the final, but I do not give you an A for the course, i.e., the case where the antecedent is true and the consequent is false. In all other cases, what I said was true. Hence, the conditional in this case is plausibly taken to be a material conditional.

Material Biconditionals

SEMANTIC RULE FOR BICONDITIONALS

A biconditional is true if the component statements on either side of the (main) double arrow have the same truth value; otherwise it is false.

Truth table schema for biconditionals

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Comment: Note that a material biconditional is logically equivalent to (i.e., roughly, means the same as) a conjunct-tion of two material conditionals, that is: $p \leftrightarrow q$ means the same as $(p \rightarrow q) \bullet (q \rightarrow p)$.

Calculating the Truth Value of a Complex Formula

We will now show how to use the truth tables for our logical operators to calculate the truth value of a complex formula from the truth values of its atomic constituents. We begin by listing all the ways those constituents can be true or false.

A	B	(~B	→	A)	↔	((A	•	B)	∨	~A)
T	T									
T	F									
F	T									
F	F									

We now start working from the inside out, calculating the truth values of the smaller "subformulas" of our complex formula when we have enough information. We will indicate in black the "subformulas" of the complex formula whose truth values we are able to calculate at each stage; those we can't will be partially greyed out. Red truth values are the ones we are calculating; yellow highlights mark the truth values involved in the calculations. Thus, because we are given the truth values for A over to the left, we can calculate the truth value for ~A in each row:

A	B	(~B	→	A)	↔	((A	•	B)	∨	~A)
T	T									F
T	F									F
F	T									T
F	F									T

We also have the truth values for B so we can also calculate the truth values for ~B:

A	B	(~B	→	A)	↔	((A	•	B)	∨	~A)
T	T	F								F
T	F	T								F
F	T	F								T
F	F	T								T

But note that having the truth values for *both* A and B also gives us enough information to calculate the truth value for the subformula (A • B), so we'll fill those values in next:

A	B	(~B	→	A)	↔	((A	•	B)	∨	~A)
T	T	F					T			F
T	F	T					F			F
F	T	F					F			T
F	F	T					F			T

Because (i) we have the truth values for A and (ii) we've calculated the truth values for $\sim B$, we can now calculate the truth values for $(\sim B \rightarrow A)$:

A	B	$(\sim B$	$\rightarrow$	$A)$	$\leftrightarrow$	$((A$	$\bullet$	$B)$	$\vee$	$\sim A)$
T	T	F	T				T			F
T	F	T	T				F			F
F	T	F	T				F			T
F	F	T	F				F			T

And now that we have calculated the truth values for $\sim A$ and $(A \bullet B)$ we have enough information to calculate the truth values for $((A \bullet B) \vee \sim A)$:

A	B	$(\sim B$	$\rightarrow$	$A)$	$\leftrightarrow$	$((A$	$\bullet$	$B)$	$\vee$	$\sim A)$
T	T	F	T				T		T	F
T	F	T	T				F		F	F
F	T	F	T				F		T	T
F	F	T	F				F		T	T

We now have the truth values for the formulas $(\sim B \rightarrow A)$ and $((A \bullet B) \vee \sim A)$ occurring to the left and the right (respectively) of our complex formula. So all that is left for us to do is to calculate the truth value for that formula in each row, which we will display under the *main logical operator* $\leftrightarrow$:

A	B	$(\sim B$	$\rightarrow$	$A)$	$\leftrightarrow$	$((A$	$\bullet$	$B)$	$\vee$	$\sim A)$
T	T	F	T		T		T		T	F
T	F	T	T		F		F		F	F
F	T	F	T		T		F		T	T
F	F	T	F		F		F		T	T

Note that the truth values of our complex formula match exactly the truth values assigned to B. As we will discuss in more detail in §7.5, this means that our complex formula is *logically equivalent to* B; despite being much more complicated grammatically, the more complex formula carries exactly the same information as B!